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Hydromagnetic edge waves in a rotating stratified fluid

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The properties of edge waves confined by the interaction of buoyancy and Coriolis forces to the vicinity of a rigid plane boundary in a rotating, stratified, electrically conducting fluid pervaded by a magnetic field are established in some simple cases. The background shear is taken to be zero, the basic Alfvén velocity $\mathbf{V}$ and Brunt–Väisälä frequency N are assumed uniform, and all dissipative effects are taken to be vanishingly small. It is shown that waves trapped against the bounding wall can occur only if $\mathbf{V}$ is parallel to the wall. When the basic rotation vector $\mathbf{\Omega}$ is also parallel to the wall, the hydromagnetic edge waves have a higher frequency and smaller spatial extent perpendicular to the wall than their non-hydromagnetic counterparts, but more complex behaviour is found when $\mathbf{\Omega}$ possesses a component normal to the wall. There are conditions under which edge waves may exist even when the basic density stratification is top-heavy (i.e. when $N^2 < 0$).

1. Introduction

The propagation of hydromagnetic waves in planetary and stellar interiors can be strongly influenced by Coriolis forces and by buoyancy forces due to the action of gravity on density inhomogeneities (Chandrasekhar 1961, chap. 5). The extent of this influence may readily be judged from the dispersion relationship for small amplitude plane waves in an incompressible fluid of zero viscosity, thermal conductivity and electrical resistivity when (*a*) the undisturbed motion is solid-body rotation with angular velocity $\mathbf{\Omega}$ relative to an inertial frame, (*b*) the undisturbed density ρ_0 depends on the (downward) vertical co-ordinate s only and is such that the Brunt–Väisälä frequency

$$N \equiv \left\{\frac{|\mathbf{g}|}{\bar{\rho}}\frac{d\rho_0}{ds}\right\}^{\frac{1}{2}} \tag{1.1}$$

is uniform (where $\mathbf{g}$ is the acceleration due to gravity plus centripetal effects and $\bar{\rho}$ is the mean density), and (*c*) the undisturbed magnetic field $\mathbf{B}$ is such that the corresponding Alfvén velocity

$$\mathbf{V} \equiv \mathbf{B}(\mu\bar{\rho})^{-\frac{1}{2}} \tag{1.2}$$

is also uniform (where μ is the magnetic permeability). The approximate dispersion relationship (see Hide 1969*a*; Acheson & Hide 1973) between the angular frequency ω and vector wavenumber $\boldsymbol{\kappa}$ is the biquadratic equation

$$\omega^4 - \omega^2(2\omega_V^2 + \omega_N^2 + \omega_\Omega^2) + \omega_V^2(\omega_N^2 + \omega_V^2) = 0, \tag{1.3a}$$

with solutions

$$\omega^2 = \omega_V^2 + \tfrac{1}{2}(\omega_N^2 + \omega_\Omega^2)\{1 \pm [1 + 4\omega_V^2\omega_\Omega^2/(\omega_N^2 + \omega_\Omega^2)^2]^{\frac{1}{2}}\}, \tag{1.3b}$$

where

$$\omega_V^2 = (\mathbf{V}.\boldsymbol{\kappa})^2, \quad \omega_\Omega^2 \equiv (2\boldsymbol{\Omega}.\boldsymbol{\kappa})^2/\kappa^2, \quad \omega_N^2 \equiv (\mathbf{N}\times\boldsymbol{\kappa})^2/\kappa^2 \tag{1.4}$$

if $\mathbf{N} \equiv -N\mathbf{g}/|\mathbf{g}|$, N being real or imaginary according as the basic density stratification is bottom-heavy or top-heavy. When, as in many systems of interest, ω_V does not greatly exceed ω_Ω and ω_N in magnitude, the waves differ markedly from the familiar non-dispersive plane-polarized Alfvén waves, satisfying $\omega^2 = \omega_V^2$, which occur when $|\omega_\Omega|$ and $|\omega_N|$ are negligible in comparison with $|\omega_V|$. In general, the waves are highly dispersive and individual fluid particles move in elliptical orbits parallel to the wave fronts.

Waves satisfying (1.3) with real values of all three components $(\kappa_1, \kappa_2, \kappa_3)$ of the vector wavenumber $\boldsymbol{\kappa}$, and related work on bounded systems, have been the subject of several theoretical studies (for references see Acheson & Hide 1973), but the present investigation is evidently the first discussion of the important class of motions – here termed 'hydromagnetic edge waves' – which are confined by the action of Coriolis forces to the vicinity of a bounding surface, the component of $\boldsymbol{\kappa}$ perpendicular to the boundary being complex and with imaginary part of the appropriate sign to ensure the vanishing of the wave amplitude at great distances from the surface. Individual fluid particles oscillate along straight lines parallel to the wall, the presence of which enables the associated Coriolis force to be balanced by pressure gradients and buoyancy forces perpendicular to the wall [see equations (3.3), (3.4), (4.2) and (4.3)]. These edge waves are analogous to but more complicated than the familiar Kelvin waves (arising in the special case when $\mathbf{V} = 0$ and $\boldsymbol{\Omega}$ and $\mathbf{g}$ are parallel to each other and to the wall), which propagate with angular frequency

$$\omega = (\mathbf{N}\times\mathbf{k})/|\mathbf{k}|, \tag{1.5}$$

where $\mathbf{k}$ is the component of $\boldsymbol{\kappa}$ parallel to the wall. The Kelvin-wave phase speed is such that the vector $\omega\mathbf{k}\times\boldsymbol{\Omega}$ is directed away from the fluid, and the amplitude decays as

$$\exp\{-2\Omega|\mathbf{k}|x_3/N\} \tag{1.6}$$

with distance x_3 from the wall.

The discovery of hydromagnetic planetary waves characteristic of hydromagnetic oscillations of a rotating spherical shell and the suggestion that such waves occur in the earth's liquid core and propagate westward relative to the core material (Hide 1966), thus contributing to the general westward drift of the earth's magnetic field, have led to several attempts to identify selection mechanisms that may determine the sense of wave propagation in a variety of cylindrical and spherical rotating systems (see Hide & Stewartson 1972; Acheson & Hide 1973; Roberts & Soward 1972). The present investigation was thus undertaken not only for its general theoretical interest in the study of hydromagnetic processes in rotating fluids but also because it held out the prospect of providing further insight into selection mechanisms. Not anticipated before the detailed analysis was carried out was the finding (see §4) that buoyancy forces play an essential role in the dynamics of edge waves in the system considered here (namely a semi-infinite fluid bounded by a plane wall) even when their properties are highly modified by hydromagnetic effects.

2. Basic equations

Referred to axes rotating with constant angular velocity $\mathbf{\Omega}$, the linearized equations of motion of an incompressible, stratified, conducting fluid in a system permeated by a uniform magnetic field $\mathbf{B}$ in the undisturbed state are (under the Boussinesq approximation)

$$\partial\mathbf{u}/\partial t + 2\mathbf{\Omega}\times\mathbf{u} + \nabla p = \theta\mathbf{n} + (\mu\bar{\rho})^{-1}\,\mathbf{B}.\nabla\mathbf{b} + \nu\nabla^2\mathbf{u}, \tag{2.1}$$

$$\partial\theta/\partial t + N^2\mathbf{n}.\mathbf{u} = K\nabla^2\theta, \tag{2.2}$$

$$\partial\mathbf{b}/\partial t = \mathbf{B}.\nabla\mathbf{u} + \eta\nabla^2\mathbf{b}, \tag{2.3}$$

$$\nabla.\mathbf{u} = 0, \quad \nabla.\mathbf{b} = 0. \qquad (2.4), (2.5)$$

Here $\mathbf{u}(\mathbf{x},t)$ represents the departure of the velocity from uniform rotation, $\mathbf{b}(\mathbf{x},t)$ the magnetic field perturbation, and $\theta(\mathbf{x},t)$ the buoyancy perturbation, defined by

$$\theta \equiv -|\mathbf{g}|\rho/\bar{\rho},$$

where $\rho(\mathbf{x},t)$ is the departure of the density from the basic value $\rho_0(\mathbf{x})$. In (2.1), the departure of the total pressure (including a contribution $\mu^{-1}\mathbf{B}.\mathbf{b}$ from the magnetic field) from the hydrostatic value associated with $\rho_0(\mathbf{x})$ is written as $\bar{\rho}p$. All these perturbations from the basic state are assumed to be infinitesimally small, therefore justifying the use of linearized equations.

The kinematic viscosity of the fluid is denoted by ν, the thermal diffusivity by K and the magnetic diffusivity by

$$\eta \equiv (\mu\sigma)^{-1}, \tag{2.6}$$

where σ is the electrical conductivity of the fluid and μ the magnetic permeability. We shall define the 'upward vertical' direction by the unit vector $\mathbf{n} \equiv -\mathbf{g}/|\mathbf{g}|$, so that $\mathbf{N} = \mathbf{n}N$ [see (1.4)], the (constant) buoyancy frequency N being given by (1.1). By analogy with (1.2) we define the perturbation Alfvén velocity by

$$\mathbf{v} \equiv \mathbf{b}(\mu\bar{\rho})^{-\frac{1}{2}}.$$

Since we shall be dealing with wave motions trapped against a fixed rigid plane impermeable boundary of electrical conductivity σ_w (see §5), it is necessary to consider what boundary conditions are to be applied. The inviscid, perfectly conducting, thermally non-diffusive limits ($\nu \to 0$, $\eta \to 0$, $K \to 0$) will be taken; however, as shown by Stewartson (1960), in the steady homogeneous non-rotating case it is generally necessary to stipulate the behaviour of the *ratios* of the diffusivities in this limit when $\mathbf{B}$ has a component normal to the wall. In particular, he showed that if $\nu/\eta \to 0$ as ν, $\eta \to 0$ then the tangential component of the velocity can be allowed to be discontinuous at the wall but the tangential component of the magnetic field must be continuous; the converse applies if $\eta/\nu \to 0$ in the limit. Stewartson's boundary-layer analysis was extended to rotating fluids by Hide (1969*b*), who considered the case when both **B** and **Ω** possess components normal to the wall. His results show that, for steady flows, one can again take the tangential component of the velocity to be discontinuous and the tangential magnetic field continuous if $\nu/\eta \to 0$ as the diffusivities tend to zero.

In this paper, we shall consider time-dependent motions, and also include situations where **Ω** is parallel to the boundary. We shall not attempt a detailed analysis of the boundary layers and the associated 'jump' conditions on the tangential components of **u** and **b** in the non-diffusive limit. However, a scale analysis (see appendix) of the boundary-layer equations indicates that, when **B** possesses a normal component (more precisely, when the inequality (A 2) is satisfied), the jump $\Delta\mathbf{v}$ in the tangential part of **v** across the boundary layer is much less than that ($\Delta\mathbf{u}$) in the tangential part of **u**, provided that the diffusivities are allowed to tend to zero in an appropriate manner. The scale analysis shows that

$$|\Delta\mathbf{v}|/|\Delta\mathbf{u}| \sim (\nu/\eta)^{\frac{1}{2}} \quad \text{as} \quad \nu, \eta, K \to 0$$

provided that $K \sim \nu \ll \eta$ (with **B**, Ω, N kept fixed). This is true regardless of the inclination of **Ω** to the wall; the boundary layer is effectively a Hartmann layer. We note here in passing that, although the criterion $K \sim \nu \ll \eta$ is probably satisfied by the liquid core of the earth, it is introduced here simply in order to ensure that the fluid can slip relative to the boundary at $x_3 = 0$ in the limit $\nu \to 0$. Otherwise it transpires that edge-wave solutions of the kind considered here do not exist; see below.

When **B** possesses a normal component we shall therefore allow the tangential component of **u** to be discontinuous, while demanding that the tangential component of **b** be continuous (see also Skiles 1972). A matching electromagnetic field in the wall must then be found. On the other hand, no such condition appears to be necessary when **B** is parallel to the boundary (see Roberts 1967, p. 26).

Thus we use the boundary condition

$$[\mathbf{e}\times\mathbf{b}] = 0 \quad \text{when} \quad \mathbf{e}.\mathbf{B} \neq 0, \tag{2.7}$$

where square brackets denote the jump across the fluid–solid interface and **e** is a unit vector normal to the wall, pointing from solid to fluid. Faraday's Law

$$\nabla\times\mathbf{E} = -\partial\mathbf{b}/\partial t$$

(where **E** is the electric field) implies continuity of the tangential component of **E**:

$$[\mathbf{e}\times\mathbf{E}] = 0 \tag{2.8}$$

(and also implies $\partial[\mathbf{e}.\mathbf{b}]/\partial t = 0$). Since the boundary is impermeable we must require that

$$[\mathbf{e}.\mathbf{u}] = 0. \tag{2.9}$$

3. Edge waves: the non-hydromagnetic case

Wave motions in a rotating stratified fluid on an f-plane (with $\mathbf{f} \equiv 2\mathbf{\Omega} = f\mathbf{n}$) in the presence of a fixed rigid plane boundary have been considered in some detail by Rhines (1970). To set the stage for the hydromagnetic examples considered in later sections, we outline his analysis here, with the simple extension to the case when $\mathbf{f}$ and $\mathbf{n}$ are not necessarily parallel.

We take axes $Ox_1x_2x_3$, with Ox_3 directed normal to the wall and *into* the fluid and Ox_2 aligned with the direction of phase propagation parallel to the wall; thus we consider solutions of the form $\mathbf{u} = \mathrm{Re}\,[\hat{\mathbf{u}} \exp i(\boldsymbol{\kappa}.\mathbf{x} - \omega t)]$ etc., where $\boldsymbol{\kappa} = (0, \kappa_2, \kappa_3)$, and look for waves that are trapped against the wall by demanding that

$$\mathrm{Im}\,\kappa_3 > 0. \tag{3.1}$$

Throughout this paper, ω and κ_2 will be assumed real and non-zero. The boundary condition (2.9), taken with the fact that the wall is rigid and the exponential decay of the amplitude in the x_3 direction implied by (3.1), shows that u_3 must vanish everywhere. (We thus follow previous writers in seeking solutions for which the motion is everywhere parallel to the wall, but we must note the possibility of another type of solution, for which $u_3 \propto \sin \kappa_r x_3 \exp(-\kappa_i x_3)$. The discussion of this case might be of some theoretical interest but lies beyond the scope of the present paper.) Incompressibility [equation (2.4)] then requires that $u_2 = 0$, and so $\hat{\mathbf{u}} = (\hat{u}_1, 0, 0)$; the particle paths are parallel to the wall and perpendicular to the direction of phase propagation. The non-magnetic, non-dissipative versions of (2.1)–(2.4) give

$$-i\omega\hat{u}_1 = \hat{\theta} n_1, \quad f_3\hat{u} + i\kappa_2\hat{p} = \hat{\theta} n_2, \tag{3.2),(3.3}$$

$$-f_2\hat{u}_1 + i\kappa_3\hat{p} = \hat{\theta} n_3, \quad -i\omega\hat{\theta} + N^2 n_1\hat{u}_1 = 0. \tag{3.4),(3.5}$$

The consistency conditions for these equations may be written as

$$\omega = \pm N n_1 \tag{3.6}$$

and

$$\kappa_3/\kappa_2 = (N^2 n_1 n_3 + i\omega f_2)/(N^2 n_1 n_2 - i\omega f_3). \tag{3.7}$$

We observe that n_1 must be non-zero for a non-trivial solution to exist; in other words, there must be a component of gravity in the direction of particle oscillation. It should also be noted that f does not appear in the expression (3.6) for ω.

From (3.7) we find

$$\mathrm{Re}\,\kappa_3 = \kappa_2(N^4 n_1^2 n_2 n_3 - \omega^2 f_2 f_3)/(N^4 n_1^2 n_2^2 + \omega^2 f_3^2),$$

which shows that the decay of the amplitude with distance from the wall is generally oscillatory ($\mathrm{Re}\,\kappa_3 \neq 0$). However, it may be non-oscillatory in certain special cases, notably when neither the rotation nor gravity possesses a component normal to the wall ($f_3 = n_3 = 0$); the Kelvin wave is an example of such a case [see (1.6)]. A similar situation obtains when magnetic fields are present.

From (3.6) and (3.7) we may obtain the 'decay distance', defined as d_0, where

$$d_0 = \frac{1}{\mathrm{Im}\,\kappa_3} = \frac{N^2 n_2^2 + f_3^2}{\pm N\kappa_2(n_2 f_2 + n_3 f_3)}; \tag{3.8}$$

the sign is chosen in accordance with (3.1) and the same sign must be chosen in (3.6). In the case of an f-plane (i.e. $\mathbf{f} = f\mathbf{n}$) we have

$$\operatorname{Im}\kappa_3 = \pm N\kappa_2 f(n_2^2+n_3^2)/(N^2n_2^2+f_3^2).$$

This expression with (3.1) and (3.6) shows that $\operatorname{sgn}(\omega/\kappa_2) = \operatorname{sgn}(n_1 f)$. Thus, as pointed out by Rhines (1970), an examination of the geometry of the situation shows that the phase propagates to the left (right) of an observer facing up the sloping boundary when $f > 0$ ($f < 0$).

To obtain the group velocity $\mathbf{C}$, we note that the right-hand side of the dispersion relation (3.6) depends on the *direction*, but not the magnitude, of the projection $\mathbf{k}$ of the wave vector in the x_1x_2 plane. We may write (3.6) thus:

$$\omega = \pm\mathbf{p}.\mathbf{k}/|\mathbf{k}|,$$

where

$$\mathbf{p} \equiv N\mathbf{e}\times\mathbf{n}, \quad \mathbf{k} = (0, \kappa_2, 0).$$

Then

$$\mathbf{C} \equiv \frac{\partial\omega}{\partial\mathbf{k}} = \pm\frac{\mathbf{k}\times(\mathbf{p}\times\mathbf{k})}{|\mathbf{k}|^3} = \pm\left(-\frac{N\mathbf{k}.\mathbf{n}}{|\mathbf{k}|^2}, 0, 0\right) \equiv (C_1, 0, 0);$$

the group velocity, as noted by Rhines (1970), is directed parallel to the wave crests and has magnitude

$$|\omega n_2/\kappa_2 n_1|$$

and sign

$$\operatorname{sgn} C_1 = -\operatorname{sgn}[n_2(n_2f_2+n_3f_3)].$$

4. The hydromagnetic case: B parallel to wall

It is straightforward to extend the above theory to the hydromagnetic case with a basic imposed magnetic field $\mathbf{B}$ parallel to the wall, provided we make the assumption that the discontinuities in the tangential components of $\mathbf{u}$ and $\mathbf{b}$ are independent on $x_3 = 0$ (see appendix).

From the non-dissipative versions of (2.1)–(2.6) we find that

$$\hat{v}_2 = \hat{v}_3 = 0$$

and

$$-i\omega\hat{u}_1 = \hat{\theta}n_1 + iV_2\kappa_2\hat{v}_1, \tag{4.1}$$

$$f_3\hat{u}_1 + i\kappa_2\hat{p} = \hat{\theta}n_2, \quad -f_2\hat{u}_1 + i\kappa_3\hat{p} = \hat{\theta}n_3, \qquad (4.2), (4.3)$$

$$-i\omega\hat{\theta} + N^2n_1\hat{u}_1 = 0, \quad -i\omega\hat{v}_1 = iV_2\kappa_2\hat{u}. \qquad (4.4), (4.5)$$

Consistency of these equations demands that

$$\omega = \pm(N^2n_1^2 + V_2^2\kappa_2^2)^{\frac{1}{2}} \tag{4.6}$$

and

$$\kappa_3/\kappa_2 = (N^2n_1n_3 + i\omega f_2)/(N^2n_1n_2 - i\omega f_3) \tag{4.7}$$

[cf. (3.7)]. N^2n_1 must be non-zero, so as to ensure that κ_3 is not real; the presence of stratification in the direction of phase propagation is therefore essential if trapping is to occur. To understand why this should be, we examine the linearized vorticity equation, which is found from (2.1) to be

$$\partial(\nabla\times\mathbf{u})/\partial t - \mathbf{V}.\nabla(\nabla\times\mathbf{v}) = \mathbf{f}.\nabla\mathbf{u} + \nabla\theta\times\mathbf{n}, \tag{4.8}$$

when $\nu = 0$. In the present case, the x_1 component of the left-hand side of (4.8) vanishes, leaving

$$0 = (\kappa_2 f_2 + \kappa_3 f_3)\,\hat{u}_1 + (n_3\kappa_2 - n_2\kappa_3)\,\hat{\theta}.$$

This equation may also be derived by eliminating $\hat{p}$ from (4.2) and (4.3); it expresses a balance between the x_1 component of the term $-\mathbf{f}.\nabla\mathbf{u}$ representing compression of basic vortex lines and that of the buoyancy force curl, $\nabla\theta \times \mathbf{n}$. In the absence of stratification, the latter term vanishes, and the former must do likewise, requiring either that $\hat{u}_1 = 0$ or that κ_3 be real. No trapped wave can then exist.

From (4.6) and (4.7) we find

$$\operatorname{Im}\kappa_3 = \frac{\pm N^2 n_1 \kappa_2 (N^2 n_1^2 + V_2^2\kappa_2^2)^{\frac{1}{2}}(f_2 n_2 + f_3 n_3)}{N^4 n_1^2 n_2^2 + f_3^2(N^2 n_1^2 + V_2^2\kappa_2^2)},$$

the choice of sign being as in (4.6), and such as to give positive $\operatorname{Im}\kappa_3$. On an f-plane we again find that the phase propagates to the left of an observer facing upslope when $f > 0$ and vice versa.

With $\mathbf{f}$ not necessarily parallel to $\mathbf{n}$, the decay distance d_M is given by

$$d_M = \frac{N^4 n_1^2 n_2^2 + f_3^2(N^2 n_1^2 + V_2^2 k_2^2)}{N^2(N^2 n_1^2 + V_2^2\kappa_2^2)^{\frac{1}{2}}\,|n_1\kappa_2(n_2 f_2 + n_3 f_3)|}.$$

It can then be shown that, for fixed $\kappa_2, \mathbf{n}, \mathbf{f}$ and N,

$$\frac{d_M}{d_0} \gtrless 1 \quad \text{as} \quad \frac{f_3^2}{N^2 n_2^2}\left(1 + \frac{V_2^2\kappa_2^2}{N^2 n_1^2}\right)^{\frac{1}{2}} \gtrless 1.$$

In particular, if $f_3 = 0$ (so that the rotation vector is parallel to the wall) but $n_2 \neq 0$ (so that $|\kappa_3/\kappa_2|$ is bounded) the hydromagnetic waves are more strongly trapped than their non-hydromagnetic counterparts. If f_3 is non-zero, then the shorter hydromagnetic waves, namely those satisfying the inequality

$$\kappa_2^2 > \frac{N^2 n_1^2}{V_2^2}\left(\frac{N^4 n_2^4}{f_3^4} - 1\right), \tag{4.9}$$

are less strongly trapped than the corresponding non-hydromagnetic waves. It is clear that every trapped hydromagnetic wave satisfies condition (4.9) when $f_3^2 \geqslant N^2 n_2^2$.

Using the notation of § 3, we can write (4.6) as

$$\omega^2 = (\mathbf{p}.\mathbf{k})^2/|\mathbf{k}|^2 + (\mathbf{V}.\mathbf{k})^2;$$

hence the group velocity has components

$$C_1 \equiv \frac{\partial\omega}{\partial k_1} = -\frac{N^2 n_1 n_2}{\omega\kappa_2} + \frac{V_1 V_2 \kappa_2}{\omega}$$

and

$$C_2 \equiv \frac{\partial\omega}{\partial k_2} = \frac{V_2^2\kappa_2}{\omega}.$$

We note that $\mathbf{C}$ is not parallel to the wave crests when $V_2 \neq 0$.

5. The hydromagnetic case: B not parallel to wall

The situation is more complicated if the basic uniform magnetic field **B** permeating the whole system possesses a component perpendicular to the wall, for we are then not generally justified in taking both u_1 and v_1 to be discontinuous at the wall. As mentioned in § 2, a scale analysis of the boundary-layer equations suggests that, under appropriate assumptions about the ratios of the diffusivities as they tend to zero, we may take the tangential component of **b** to be continuous but the tangential component of **u** discontinuous across the interface. It is then necessary to consider the electromagnetic field within the wall; for definiteness, we take the latter to be of infinite thickness in the $-x_3$ direction.

It transpires that no edge wave of steady amplitude can exist under these conditions; this fact may be shown most easily by a *reductio ad absurdum* argument. We *assume* that an edge wave with exponential variation $\exp i(\kappa_2 x_2 + \kappa_3 x_3 - \omega t)$, with κ_2 and ω real and $\text{Im}\, \kappa_3 > 0$, exists in the fluid. As in the previous section, it follows that $\hat{\mathbf{u}}$ and $\hat{\mathbf{b}}$ possess only x_1 components, which, by the non-dissipative version of the induction equation (2.3), satisfy

$$-i\omega \hat{v}_1 = i(V_2 \kappa_2 + V_3 \kappa_3)\, \hat{u}_1. \tag{5.1}$$

Since the fluid is perfectly conducting, the electric field **E** is given by

$$\mathbf{E} = -\mathbf{u} \times (\mathbf{B} + \mathbf{b}),$$

which is $\simeq -\mathbf{u} \times \mathbf{B}$ on linearized theory. Hence the tangential component of **E** at the interface is

$$\mathbf{E}_{\tan} \equiv (0, u_1 B_3, 0).$$

If the wall is of finite conductivity σ_w, the magnetic field perturbation within the wall satisfies

$$\partial \mathbf{b}/\partial t = \eta_w \nabla^2 \mathbf{b}, \quad \eta_w \equiv (\mu \sigma_w)^{-1}$$

and the electric field is given by

$$\mathbf{E} = \eta_w \nabla \times \mathbf{b}.$$

It is now easy to show that the continuity conditions on **b** and $\mathbf{E}_{\tan}$ across the interface and (5.1) cannot all be satisfied simultaneously; thus a contradiction is reached, and the postulated steady edge wave cannot exist.

The perfectly conducting wall can be regarded as the limit of the above, with $\eta_w \to 0$. The magnetic field in the wall vanishes, except in a surface layer 'in' the wall, where a surface current takes up the discontinuity in the tangential component of **b**. However, **E** vanishes everywhere in the wall, and so cannot match $\mathbf{E}_{\tan}$ in the fluid.

When the wall is an insulator ($\eta_w^{-1} \to 0$), the magnetic field within it must be curl-free. However, this field cannot have both an x_1 component and the required (x_2, t) dependence; edge waves are again ruled out.

6. Top-heavy density stratification

Finally, we describe a simple example in which the presence of a magnetic field may allow the existence of trapped waves of steady amplitude even when the basic density distribution is 'top-heavy', i.e. $N^2 < 0$ [see (1.1)].

We consider a fluid domain bounded above and below by rigid horizontal boundaries and laterally by a vertical wall. It is convenient here to use axes $Oxyz$ with Oz vertical and Oy normal to the wall and pointing into the fluid. The horizontal boundaries are given by

$$z = 0, H, \quad -\infty < x < \infty, \quad y \geqslant 0$$

and the vertical wall by

$$0 \leqslant z \leqslant H, \quad -\infty < x < \infty, \quad y = 0.$$

We take the basic Alfvén velocity as $(V, 0, 0)$ and the basic rotation vector as $(0, 0, \frac{1}{2}f)$, $f > 0$; then, with $\genfrac{}{}{0pt}{}{\sin}{\cos}(mz) \exp [i(kx - \omega t) - \lambda y]$ dependence, we obtain, as in §4,

$$\omega^2 = N^2k^2/(k^2+m^2) + V^2k^2, \tag{6.1}$$

$$\lambda = \omega f(k^2+m^2)/kN^2, \tag{6.2}$$

where the upper and lower boundary conditions require

$$m = l\pi/H, \quad l = 1, 2, \ldots;$$

(6.1) and (6.2) are analogous to (4.6) and (4.7) respectively, written in a different co-ordinate frame.

Supposing now that $N^2 < 0$, corresponding to a basic density *increasing* with height, we see that trapped stable waves ($\lambda > 0$, ω real) may still occur, provided that

$$\omega/k < 0, \quad V^2(l^2\pi^2/H^2 + k^2) > |N^2|. \tag{6.3a, b}$$

The first of these requirements means that the phase must travel in the direction of decreasing x (i.e. opposite to the direction taken when $N^2 > 0$); the second condition holds for all real k and positive integers l if

$$V^2 > |N^2|\pi^2/H^2.$$

Thus, in the present configuration, if the basic magnetic field is sufficiently strong it overcomes the effect of the unstable density stratification, and allows the presence of trapped stable waves. No unstable (non-dissipative) modes satisfying the boundary conditions can then exist. We note that the 'quantization' of the vertical wavenumber by the presence of the horizontal boundaries is essential to the stabilization process. In the absence of such boundaries, perturbations of sufficiently small total wavenumber $(k^2+m^2)^{\frac{1}{2}}$ in the x, z plane would be unstable. (For systems constrained by magnetic fields and/or rotation, dissipative effects can give rise to instability under conditions when non-dissipative theory would predict strong stability, although the literature contains little information concerning the dependence of growth rates on the dissipative coefficients. Such instabilities would in general be characterized by non-zero values of the real part of ω and might therefore contribute to the generation of edge waves.)

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Appendix. Scale analysis of boundary-layer equations

When $\mathbf{V}$ possesses a significant normal component (in a sense to be made more precise below) we assume the existence of a boundary layer of thickness δ in the x_3 direction. If the flow has a length scale L in the x_1, x_2 plane ($L \gg \delta$) it then follows from $\nabla . \mathbf{u} = \nabla . \mathbf{v} = 0$ that

$$\Delta u_3/\delta \sim \Delta u_1/L, \quad \Delta v_3/\delta \sim \Delta v_1/L, \tag{A 1a, b}$$

where Δu_1 is a typical magnitude of the variation in u_1 and u_2 across the boundary layer, Δu_3 represents the corresponding variation in u_3, and so on.

As in the main part of this paper, we assume infinitesimal disturbances, so that nonlinear terms may be neglected. We can say that V_3 leads to a significant contribution to the operator $\mathbf{V} . \nabla$ [see (1.2), (2.1) and (2.3)] if

$$V_3 \gtrsim V_1 \delta/L; \tag{A 2}$$

for convenience, we postulate

$$\kappa \sim \nu, \quad \nu/\eta \to 0 \quad \text{as} \quad \eta \to 0. \tag{A 3}$$

We initially assume that time derivatives are negligible in the boundary-layer equations; self-consistency of this assumption will be checked *a posteriori*. The x_1 component of the steady induction equation in the boundary layer is

$$\mathbf{V} . \nabla u_1 = -\eta\, \partial^2 v_1/\partial x_3^2,$$

which, using (A 2), gives

$$\Delta v_1/\Delta u_1 \sim V_3 \delta/\eta. \tag{A 4}$$

Similarly, the steady buoyancy equation is

$$N^2 \mathbf{n} . \mathbf{u} = K\, \partial^2 \theta/\partial x_3^2,$$

which yields

$$\Delta \theta/\Delta u_1 \lesssim N^2 \delta^2/K. \tag{A 5}$$

The x_1 component of the steady momentum equation is

$$\partial p/\partial x_1 + 2(\Omega_2 u_3 - \Omega_3 u_2) - n_1 \theta = \mathbf{V} . \nabla v_1 + \nu\, \partial^2 u_1/\partial x_3^2, \tag{A 6}$$

and we shall suppose that the dominant terms are those on the right-hand side (corresponding to a 'Hartmann balance'). Using (A 4) it follows that the boundary-layer thickness δ has the Hartmann value

$$(\eta\nu)^{\frac{1}{2}}/V_3 \tag{A 7}$$

and we can now check with the aid of the x_3-momentum equation and (A 1 *a*), (A 5) and (A 7) that all the terms on the left of (A 6), as well as the time derivatives in the equations, are indeed negligible under (A 3).

Expressions (A 4) and (A 7) yield

$$\Delta v_1/\Delta u_1 \sim (\nu/\eta)^{\frac{1}{2}}, \tag{A 8}$$

which, by (A 3), tends to zero with η. It therefore seems justifiable in the inviscid, thermally non-diffusive, perfectly conducting limit specified by (A 3) to take the tangential component of $\mathbf{u}$ as discontinuous ($\Delta u_1 \neq 0$) but the tangential component of $\mathbf{v}$ as continuous ($\Delta v_1 = 0$) at the wall. This argument is not entirely rigorous, but confidence in its applicability to the general case studied here is strengthened by the consistency between (A 8) and the solutions of Stewartson (1960) and Hide (1969*b*) for more restrictive circumstances.

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